Independence on Triangular Triangle Boards

Dietrich, Heiko
Harborth, Heiko

Veröffentlicht in:
Abhandlungen der Braunschweigischen Wissenschaftlichen Gesellschaft Band 54, 2004, S.73-84

J. Cramer Verlag, Braunschweig

Abhandlungen der BWG · **54**: 73–87 · Braunschweig, Mai 2005

Independence on Triangular Triangle* Boards

HEIKO DIETRICH & HEIKO HARBORTH

Diskrete Mathematik
Technische Universität Braunschweig
38023 Braunschweig, Germany
e-mail: h.harborth@tu-bs.de

Abstract: Triangular parts of the Euclidean triangle tessellation of the plane are considered as gameboards T_n. Thirteen chess-like pieces are defined as in [1]. The independence number β_n is the maximum number of non-attacking copies of a piece on T_n. For nine of the chess-like pieces β_n is determined completely.

1. Introduction

The independence number β of a graph is the maximum number of nonadjacent vertices. For a chess-like piece P on a gameboard B the P-graph has the cells of B as its vertices and edges are between cells which are at a distance of one move of P from one another.

Here we consider triangle boards $B = T_n$ which are equilateral triangles of n^2 triangles of the Euclidean triangle tessellation of the plane such that the sides of T_n consist of n sides of the triangle cells (see Figure 1).

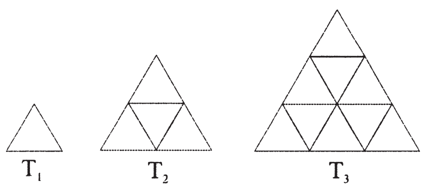

Figure 1: Gameboards.

* Eingegangen 14.04.2003

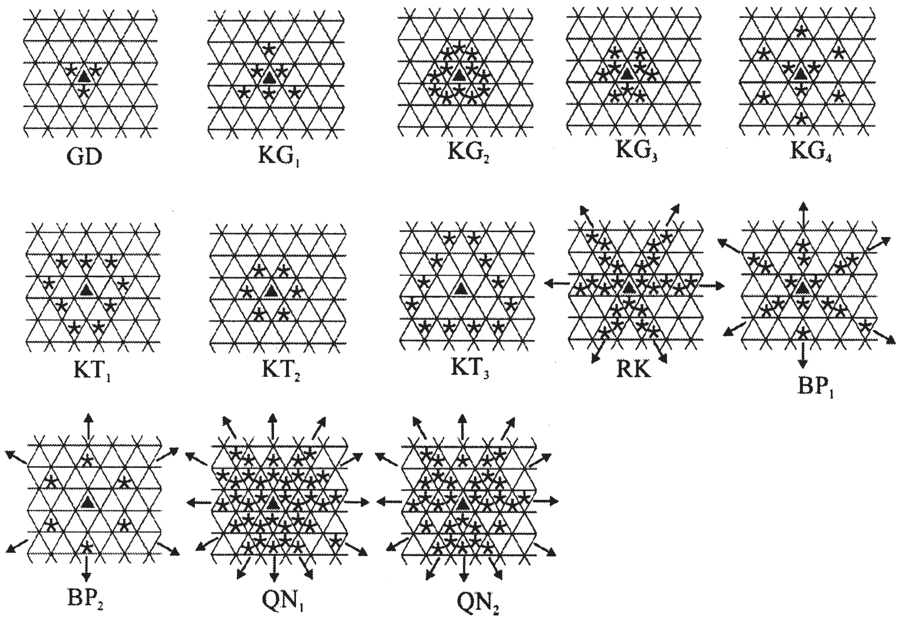

Figure 2: Moves of chesspieces on T_n.

In [1] thirteen chess-like pieces P are defined as depicted in Figure 2, where a piece P attacks all triangles indicated by a star. For most of these chess-like pieces P we will determine the independence numbers $\beta_n(P)$ with respect to T_n.

In [4, 5] the independence numbers are given for the graphs of three chesslike pieces on the corresponding triangular hexagon boards.

2. Grid

A grid (GD) attacks all edge-to-edge neighboring triangles.

Theorem 1. The independence number of GD is $\beta_n(\mathrm{GD}) = \binom{n+1}{2}$.

Proof: In the 2-coloring of the triangles such that triangles with a common edge are of different color, any grid attacks only triangles of the other color. Thus β_n is at least $\binom{n+1}{2}$, the larger number of triangles of the two color classes.

To see that at most $\binom{n+1}{2}$ independent grids are possible we use a tessellation of T_n by tiles T_1 and D as in Figure 3, and we observe that each tile contains at most one independent grid. □

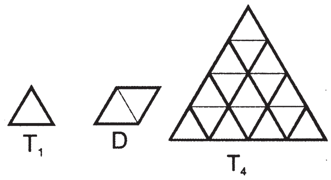

Figure 3: Tessellation of T_4 by tiles D and T_1.

3. King 1

In addition to the triangles attacked by a grid, king 1 (KG_1) attacks the vertex-to-vertex neighboring triangles on the three diagonals.

Theorem 2. The independence number of KG_1 is $\beta_n(KG_1) = \binom{n+1}{2}$.

Proof: Since KG_1 attacks only triangles of the other color in the 2-coloring as in the grid case, we obtain $\beta_n \geq \binom{n+1}{2}$ analogously. For $\beta_n \leq \binom{n+1}{2}$ the argument of the grid case works correspondingly. □

4. King 2

A king 2 (KG_2) attacks all neighboring triangles.

Theorem 3. The independence number of KG_2 is

$$\beta_n(KG_2) = \begin{cases} \frac{1}{3}\binom{n+2}{2} - 1 & \text{if } n \equiv 2,4,5,7 \pmod{12}, \\ \left\lfloor \frac{1}{3}\binom{n+2}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof: The independence number β_n is equivalent to the maximum number of disjoint triangles which can be chosen from T_n (see Figure 4).

Using regular hexagons instead of the vertex points of the triangles, it is proved in [2] that a partition into triangular triples of hexagons is possible if and only

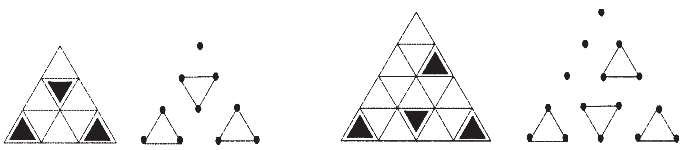

Figure 4: Maximum number of KG_2s on T_3 and T_4.

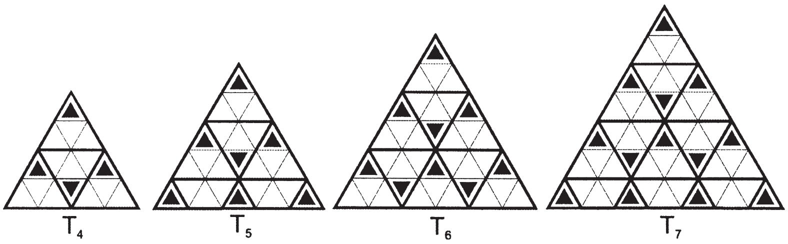

Figure 5: Tessellations of $T_4,\ldots,T_7$ by tiles T_2 with independent KG_3s.

if $n \equiv 1, 8, 10, 11 \pmod{12}$. Thus, in these residue classes we have $\beta_n = \frac{1}{3}\binom{n+2}{2}$, that is, one third of the number of vertex points of T_n. From the constructions in [5] it follows that $\beta_n = \frac{1}{3}\binom{n+2}{2} - 1$ if $n \equiv 2, 4, 5, 7 \pmod{12}$ and $\beta_n = \frac{1}{3}[\binom{n+2}{2} - 1]$ if $n \equiv 0 \pmod 3$. □

5. King 3

A king 3 (KG_3) attacks all neighboring triangles except the vertex-to-vertex neighboring triangles on the three diagonals.

Theorem 4. The independence number of KG_3 is $\beta_n(KG_3) = \left\lceil \frac{n^2}{4} \right\rceil$.

Proof: For even n a tessellation of T_n by $\frac{n^2}{4}$ tiles T_2 is possible (see Figure 5). Since every T_2 contains at most one independent KG_3 we have $\beta_n \le \frac{n^2}{4}$. At least $\frac{n^2}{4}$ independent KG_3s can be placed as it is shown in Figure 5 where the single triangles in the horizontal rows of the T_2s are used. Thus the proof for $n \equiv 0 \pmod 2$ is complete.

For odd n we obtain $\beta_n \ge \left\lceil \frac{n^2}{4} \right\rceil$ from $\frac{(n-1)^2}{4}$ kings KG_3 on T_{n-1} as before.

Then $\frac{n+1}{2}$ kings KG_3 are added in the last row of T_n (see Figure 5).

For $n \equiv 1 \pmod 4$ three boards $T_{\frac{n-1}{2}}$ at the corners of T_n are tessellated by T_2s. The remaining part is contained in $T_{\frac{n+3}{2}}$ (see Figure 6 for $n = 9$).

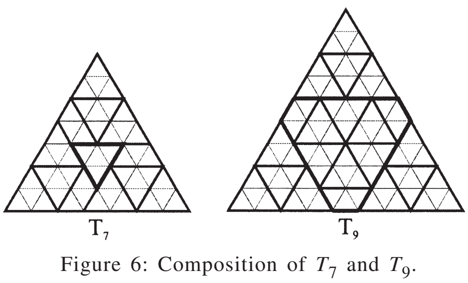

Figure 6: Composition of T_7 and T_9.

Altogether at most $3\,\frac{(n-1)^2}{16} + \frac{(n+3)^2}{16} = \left\lceil \frac{n^2}{4} \right\rceil$ independent KG_3s are possible on T_n.

For $n \equiv 3 (\mathrm{mod}\ 4)$ we use three boards $T_{\frac{n+1}{2}}$ at the corners of T_n having pairwise one vertex triangle in common. The remaining part is a $T_{\frac{n-3}{2}}$ (see Figure 6 for $n = 7$). Then there are at most $3\,\frac{(n+1)^2}{16} + \frac{(n-3)^2}{16} = \left\lceil \frac{n^2}{4} \right\rceil$ kings KG_3. □

6. King 4

In addition to the triangles attacked by a grid, king 4 (KG_4) attacks the two translated neighboring triangles in the three diagonal directions.

Theorem 5. The independence number of KG_4 is

$$\beta_n(KG_4) = \begin{cases} \frac{n^2+n}{3} + 1 & \text{if } n \equiv 2 (\mathrm{mod}\ 3), \\ \left\lceil \frac{n^2+n}{3} \right\rceil & \text{otherwise.} \end{cases}$$

Proof: There are 1, 3, and 4 independent KG_4s possible on T_1, T_2, and T_3, respectively, as in the top parts of T_4, T_5, and T_6 in Figure 7. Then T_{n-3} can be completed by $2n$–2 independent KG_4s in the last three rows of T_n as shown in Figure 7. Thus by induction the asserted terms are lower bounds for β_n.

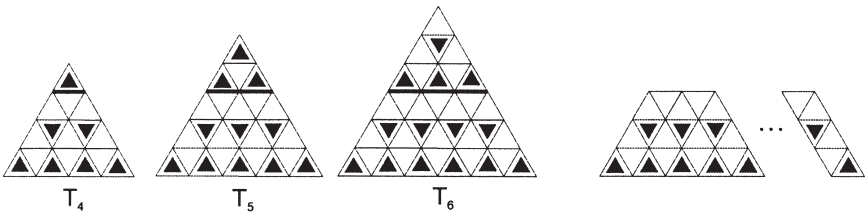

Figure 7: Gameboards T_4, T_5, T_6, and three additional rows of triangles.

To prove the upper bound we observe that there are at most 1, 3, and 4 independent KG_4s on gameboards T_1, T_2, and T_3, respectively.

If at most $2n - 2$ independent KG_4s can occur in the last three rows of T_n then the asserted terms are upper bounds by induction. However, this works only for $n \equiv 0 (\mathrm{mod}\ 3)$.

For $n \equiv 0 (\mathrm{mod}\ 3)$ we partition the last three rows of T_n into tiles A, B, and $\frac{n-3}{3}$ tiles P (see Figure 8). Then we partition A, B, and P into singles, pairs, and triples of dependent triangles as in Figure 8. Altogether, we have at most $2n - 1$ independent

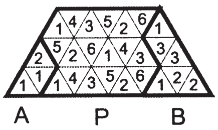

Figure 8: Partition into tiles A, B, and P.

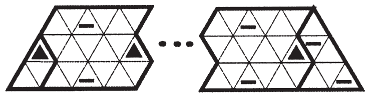

Figure 9: The forced KG_4s.

KG_4s. If $2n - 1$ independent KG_4s occur then one KG_4 has to be on the single triangle in A. Then one KG_4 is forced in every P so that B contains at most two KG_4s (see Figure 9). Thus $\beta \leq 2n - 2$ for the last three rows and induction mod 3 can be used.

For $n \equiv 1 \pmod 3$ we consider the partition into singles, pairs, and triples as in Figure 10. Again we have at most $2n - 1$ independent KG_4s. If the number of KG_4s is exactly $2n - 1$ then one KG_4 has to be on the single triangle (6) of B. This KG_4 forces successively all KG_4s as indicated in Figure 11. Now these forced KG_4s block in row $n - 3$ all triangles with a minus sign in Figure 11 and thus at most $\frac{2}{3}(n - 4)$ triangles remain for possible KG_4s. Thus the last four rows contain at most $2n - 1 + \frac{2}{3}(n - 4)$ independent KG_4s. Then induction from $n - 4$ to n works if exactly $2n - 1$ independent KG_4s are in the last three rows and otherwise we can use induction from $n - 3$ to n.

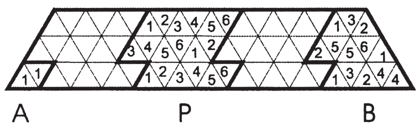

Figure 10: Partition into tiles A, B, and P.

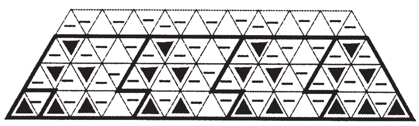

Figure 11: The forced KG_4s.

For $n \equiv 2 \pmod 3$ we partition the last three rows of T_n into tiles A, B, Q, and $\frac{n-5}{3}$ tiles P. By partition into singles, pairs and triples of dependent triangles it follows that at most $2n - 1$ independent KG_4s are possible (see Figure 12). Assuming that there are exactly $2n - 1$ independent KG_4s, two KG_4s have to be in B as in Figure 13.

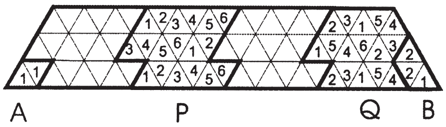

Figure 12: Partition into tiles A, B, Q, and P.

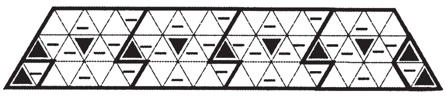

Figure 13: The forced KG_4s.

The upper one forces all leftmost KG_4s in the tiles P and one KG_4 in tile A. One KG_4 has to be on the single triangle (6) in Q forcing a second KG_4 in every P as in Figure 13. Now we consider the last row where at most $2 + 2\frac{n-2}{3}$ independent KG_4s remain possible since there are two neighboring pairs of free triangles in the tiles P and in Q. Thus induction from $n - 1$ to n works if exactly $2n - 1$ independent KG_4s occur in the last three rows, and otherwise induction from $n - 3$ to n can be used. □

7. Knight 1

A knight 1 (KT_1) moves edge-to-edge two triangles in one direction and then attacks the left and right neighboring triangle.

Theorem 6. The independence number of KT_1 is $\beta_n(KT_1) = \binom{n+1}{2}$ with the exception $\beta_2(KT_1) = 4$.

Proof: Since KT_1 attacks only triangles of the other color in the unique 2-coloring as in the grid case, we obtain $\beta_n \geq \binom{n+1}{2}$. Using induction from n to $n + 1$ we prove that there are at most $3(n - 1)$ independent KT_1s in the last three rows of triangles of T_n if $n \geq 3$ (see Figure 14).

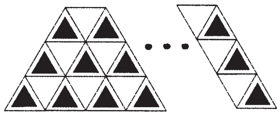

Figure 14: Maximum number of KT_1s in the last three rows of T_n.

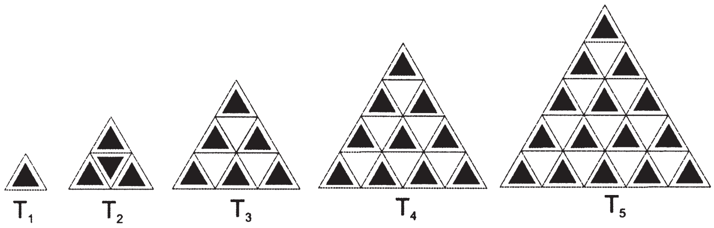

Figure 15: Maximum number of KT_1s on $T_1,\ldots,T_5$.

The maximum numbers of KT_1s on $T_1, \ldots, T_5$ are shown in Figure 15. Induction from n to $n + 3$ for $n \geq 3$ using the additional three rows in Figure 14 proves $\beta_n \leq \binom{n+1}{2}$. □

8. Knight 2

A knight 2 (KT_2) attacks all edge-to-edge neighbors of the neighboring triangles on the diagonals or, equivalently, it moves one triangle in a straight direction and then one triangle in a diagonal direction.

Theorem 7. The independence number of KT_2 is $\beta_n(KT_2) = \left\lceil \frac{n^2}{3} \right\rceil$ with the exceptions $\beta_3(KT_2) = 4$, $\beta_4(KT_2) = 7$, $\beta_5(KT_2) = 10$, and $\beta_6(KT_2) = 13$.

Proof: In the unique 2-coloring of the triangles, a KT_2 attacks only triangles of the same color and the two color classes can be discussed seperately.

Using regular hexagons instead of the upwards or downwards oriented triangles (see Figure 16) we obtain triangular hexagon boards H_n or H_{n-1}, respectively, as in [5]. Then the moves of KT_2 are equivalent to the moves of king 1 in [5], where the independence number is proved to be $\left\lfloor \frac{n(n+1)+4}{6} \right\rfloor$ with the exceptions 3 for $n = 3$ and 6 for $n = 5$. Together we obtain $\beta_n(KT_2) = \left\lfloor \frac{n(n+1)+4}{6} \right\rfloor + \left\lfloor \frac{(n-1)n+4}{6} \right\rfloor = \left\lceil \frac{n^2}{3} \right\rceil$ with the exceptions $\beta_3 = 4$, $\beta_4 = 7$, $\beta_5 = 10$, and $\beta_6 = 13$. □

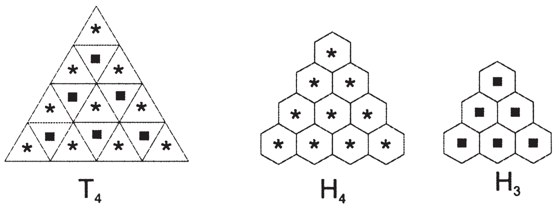

Figure 16: Splitting the two color classes of T_4 into triangular hexagon boards H_4 and H_3.

9. Knight 3

A knight 3 (KT_3) attacks all vertex-to-vertex neighboring triangles of its three vertex-to-vertex neighboring triangles on the diagonals.

Theorem 8. The independence number of KT_3 is $\beta_n(KT_3) = \binom{n+1}{2}$ with the exception $\beta_2(KT_3) = 4$.

Proof: Since KT_3 attacks only triangles of the other color in the unique 2-coloring of the triangles it follows $\beta_n \geq \binom{n+1}{2}$. For $\beta_n \leq \binom{n+1}{2}$ the arguments of the knight 1 case work correspondingly. □

10. Rook

A rook (RK) attacks all triangles along its three moving lines which are depicted in Figure 2.

Theorem 9. The independence number of RK is $\beta_n(\text{RK}) = \left\lfloor \frac{2n+1}{3} \right\rfloor$.

Proof: For $n \equiv 0, 1 (\text{mod } 3)$ we start with one rook on the first triangle of the $\left\lfloor \frac{n+3}{3} \right\rfloor$th row of T_n. Then row by row we place one rook on the first possible upwards oriented triangle cyclically to the right of the preceding rook (see Figure 17).

Altogether there are $n - \left\lfloor \frac{n+3}{3} \right\rfloor + 1 = \left\lfloor \frac{2n+1}{3} \right\rfloor$ independent rooks possible on T_n. For $n \equiv 2 (\text{mod } 3)$ we use the gameboard T_{n-1} with $\left\lfloor \frac{2(n-1)+1}{3} \right\rfloor = \left\lfloor \frac{2n+1}{3} \right\rfloor$ independent rooks.

To show $\beta_n \leq \left\lfloor \frac{2n+1}{3} \right\rfloor$ we observe that any rook on an upwards oriented triangle attacks $4n - 4$ triangles where the three neighboring triangles are attacked twice. Any rook on a downwards oriented triangle attacks $4n - 2$ triangles. This can be proved inductively since the addition of a row enlarges the number of attacked triangles by four (see Figur 18). If there are r_1 rooks on upwards oriented and r_2 rooks on downwards oriented triangles then there are $r = r_1 + r_2$ rooks attacking $r_1(4n - 4) + r_2(4n - 2) = 4nr - 4r + 2r_2$ triangles. The number of triangles attacked

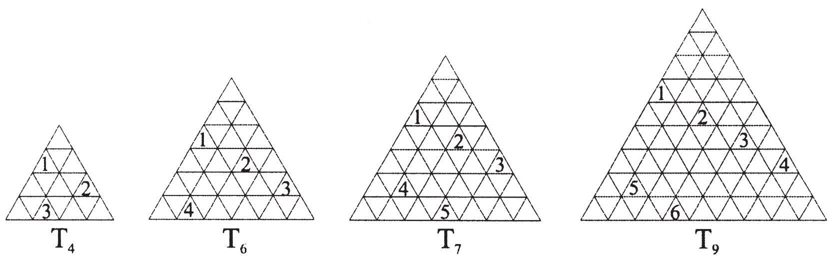

Figure 17: The sequence of placing RKs on T_n with $n \equiv 0,1 (\text{mod } 3)$.

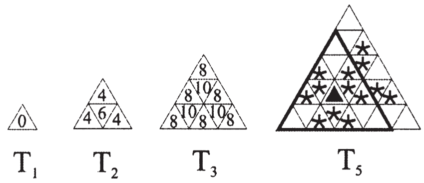

Figure 18: Numbers of attacked triangles.

by all r rooks in one of the three directions is $(2n-2i_1)+(2n-2i_2)+\ldots+(2n-2i_r)$ with $1 \le i_1 < i_2 < \ldots < i_r \le n$, that is, at most $2nr - r(r+1)$. It follows $4r(n-1) \le 4nr - 4r + 2r_2 \le 3(2nr - r(r+1))$ which is equivalent to $r \le \frac{2n+1}{3}$. □

11. Bishop 1

A bishop 1 (BP_1) attacks all triangles along its three diagonal moving lines (see Figure 2).

Theorem 10. For the independence number of BP_1 we have

$$\beta_n(BP_1) = 2n-3 \quad \text{if } n \equiv 0 (\text{mod } 3),$$

$$\beta_n(BP_1) = 2n-1 \quad \text{if } n = \tfrac{3^k+1}{2},\ k \ge 0,$$

$$2n-3 \le \beta_n(BP_1) \le 2n-1 \quad \text{otherwise.}$$

Proof: To see $\beta_n \ge 2n-3$ we place one BP_1 in the triangle at the top. Then starting with the third and fourth triangle at the left and the right border, respectively, a BP_1 is placed in every third triangle. In the inner part of the last row every third triangle remains for another independent BP_1 (see Figure 19). Since the first or second column from both sides are free of a BP_1 we obtain $\beta_n \ge 2n-3$ and by the pigeon-hole principle $\beta_n \le 2n-1$.

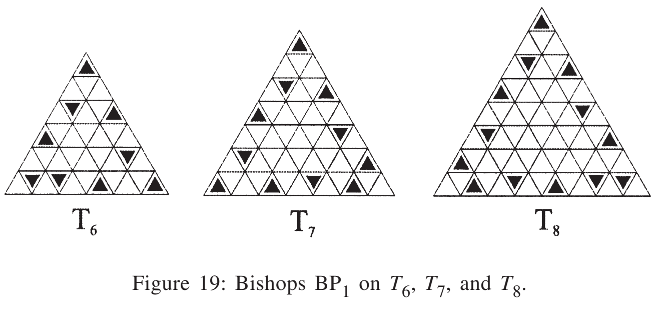

Figure 19: Bishops BP_1 on T_6, T_7, and T_8.

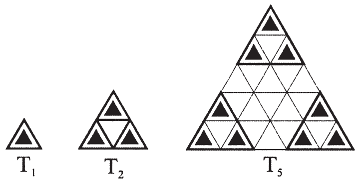

Figure 20: The gameboards $T_{\frac{3^k+1}{2}}$ with $k = 0, 1, 2$.

We consider three classes of weighted parallel lines of T_n which give their weight to all triangles intersected by the line. Starting with the upper vertex triangle of T_n the weights of the parallel lines are periodically -1, -1, 2 to the right and to the left. Then the weights of the lines in the two directions of the other moving lines are determined such that for every triangle the sum of the three weights is zero (see Figure 21). Now one weight 2 belongs uniquely to every independent BP_1 so that the number of weights 2 is an upper bound for β_n. For $n \equiv 0(\mathrm{mod}\ 3)$ this number of weights 2 is $2n - 3$ and $\beta_n \leq 2n - 3$ follows.

Three copies of a board T_m with $2m - 1 = 3^k$ independent BP_1s placed into the corners of a T_n with $2n - 1 = 3^{k+1}$, inductively imply a T_n with $2n - 1$ independent BP_1s. Starting with T_1 we obtain $\beta_n \geq 2n - 1$ for $n = \frac{3^k + 1}{2}$ (see Figure 20).

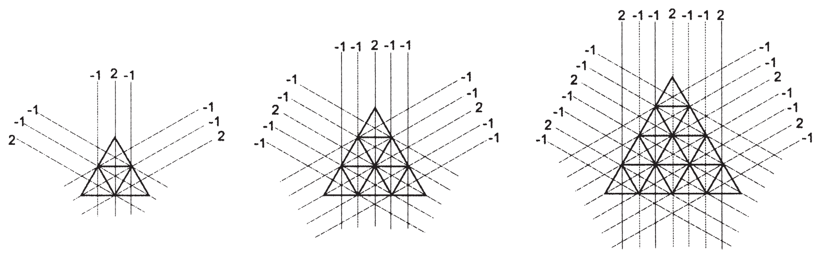

Figure 21: Weights for T_2, T_3, and T_4.

For $n \leq 12$ the values of $\beta_n(BP_1)$ not given by Theorem 10 are determined by computer (see Table 1). It remains open whether $\beta_n(BP_1) = 2n - 1$ for any n not mentioned in Theorem 10 and whether $\beta_n(BP_1) = 2n - 2$ is possible.

n	1	2	3	4	5	6	7	8	9	10	11	12
β_n	1	3	3	5	9	9	11	13	15	17	19	21

Table 1: Values of $\beta_n(BP_1)$ for $n \leq 12$.

12. Bishop 2

In the unique 2-coloring as in the grid case a bishop 2 (BP_2) attacks all triangles of the same color along its three diagonal moving lines (see Figure 2).

Theorem 11. For the independence number of BP_2 we have $\beta_1(BP_2) = 1$, $\beta_2(BP_2) =$ 4, $\beta_4(BP_2) = 8$, and

$$\begin{array}{rll} 4n-18 \le \beta_n(BP_2) \le 4n-6 & & \text{if } n \equiv 0{,}2 (\bmod 3),\ n \ge 5, \\ 4n-20 \le \beta_n(BP_2) \le 4n-8 & & \text{if } n \equiv 1 (\bmod 3),\ n \ge 7, \\ \beta_n(BP_2) = 4n-6 & & \text{if } n = \frac{3^{k+1}+3}{2},\ k \ge 0. \end{array}$$

Proof: The independence number of BP_2 can be determined as the sum of the independence numbers $\beta_n^D(BP_2)$ and $\beta_n^U(BP_2)$ for the two color classes D_n and U_n, respectively. Since D_n and U_{n-1} are of the same structure (see Figure 22) we obtain $\beta_n = \beta_{n-1}^U + \beta_n^U$. For U_n the moves of BP_2 stay within one of three disjunct color classes as in Figure 23 where the moving lines are drawn and the BP_2s are placed on the vertex points. For $n \equiv 0, 2 (\bmod 3)$ there are three boards A_n and for $n \equiv 1 (\bmod 3)$ there are two boards B_n and one board A_n. Examples for the different types of the three new boards are shown in Figure 24. Using the corresponding independence numbers $\beta_n^A(BP_2)$ and $\beta_n^B(BP_2)$ for β_n^U we obtain

$$\beta_n = \begin{cases} 3\beta_n^A + 3\beta_{n-1}^A & \text{if } n \equiv 0 (\bmod 3), \\ 2\beta_n^B + \beta_n^A + 3\beta_{n-1}^A & \text{if } n \equiv 1 (\bmod 3), \\ 3\beta_n^A + 2\,\beta_{n-1}^B + \beta_{n-1}^A & \text{if } n \equiv 2 (\bmod 3). \end{cases}$$

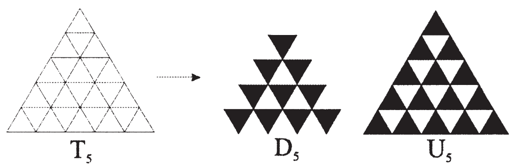

Figure 22: Partition of T_5 into its two color classes.

The following inequalities for β_n^A and β_n^B imply the inequalities of Theorem 11.

$$\begin{array}{ll} 2\frac{n}{3}-3 \le \beta_n^A \le 2\frac{n}{3}-1 & \text{if } n \equiv 0 (\bmod 3), \\ 2\,\frac{n-1}{3}-1 \le \beta_n^A \le 2\,\frac{n-1}{3}+1 & \text{if } n \equiv 1 (\bmod 3), \\ 2\,\frac{n-2}{3}-1 \le \beta_n^A \le 2\,\frac{n-2}{3}+1 & \text{if } n \equiv 2 (\bmod 3), \\ 2\,\frac{n-1}{3}-3 \le \beta_n^B \le 2\,\frac{n-1}{3}-1 & \text{if } n \equiv 1 (\bmod 3). \end{array}$$

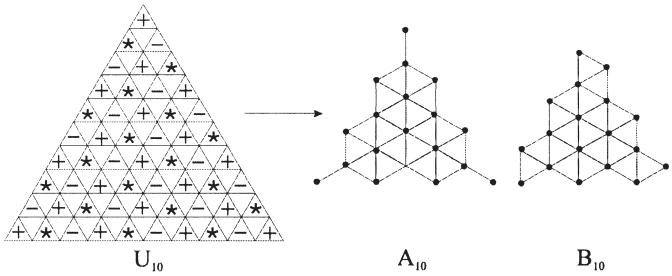

Figure 23: Partition of U_{10} into its three color classes.

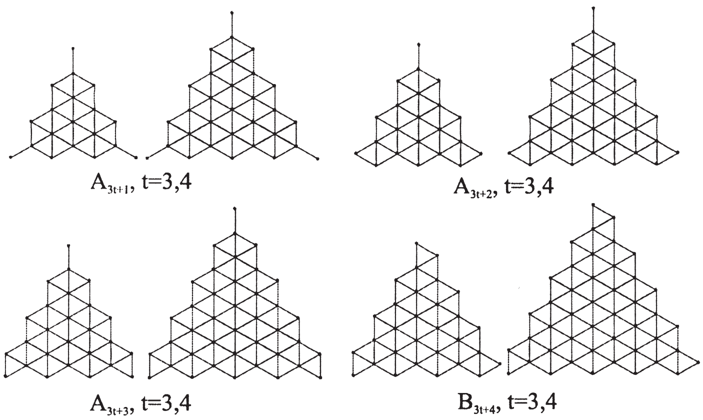

Figure 24: Examples of boards A_n and B_n.

To prove these inequalities we place BP_2s on A_n for $n \equiv 1 (\mathrm{mod}\ 3)$ as in Figure 25, that is, we choose every third vertex of the three borders of A_n in an appropriate way. Then there are $2\,\frac{n-1}{3} - 1$ independent BP_2s on A_n. Together with the fact that

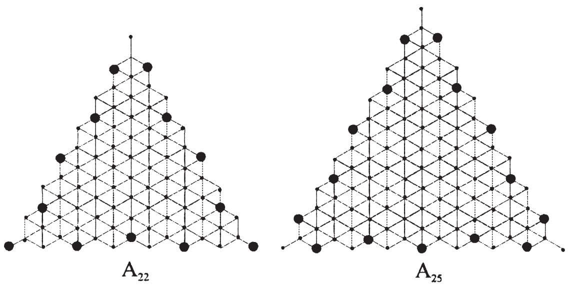

Figure 25: Independent BP_2s on A_n, $n \equiv 1 (\mathrm{mod}\ 3)$.

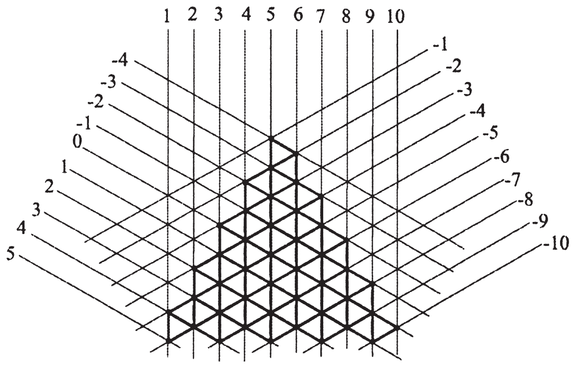

Figure 26: Weighted lines for B_{16}.

$$A_n \subset A_{n+1} \subset A_{n+2} \subset B_{n+3} \qquad \text{if } n \equiv 1 (\text{mod } 3)$$

the lower estimations of β_n^A and β_n^B are proved.

The upper estimations for β_n^A and β_n^B follow by the pigeonhole principle. The corresponding estimation for β_n^B is $\beta_n^B \leq 2\,\frac{n-1}{3}$. However, this bound can be diminished by 1 since the weights of the lines of B_n as in Figure 26 for $n = 16$ do not add up to zero. This has to be the case if every line contains exactly one BP_2.

To prove the case of equality in Theorem 11 we can place the BP_2s as it is shown in Figure 27 to achieve the upper bound $4n - 6$ for T_n with $n = \frac{3^{k+1}+3}{2}$, $k \geq 0$. □

For $n \leq 23$ the exact values of $\beta_n(BP_2)$ are listed in Table 2.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
β_n	1	4	6	8	14	18	18	21	27	30	30	36	44	50	54

n	16	17	18	19	20	21	22	23
β_n	54	60	66	66	66	72	78	81

Table 2: Exact values of $\beta_n(BP_2)$ for $n \leq 23$.

13. Queens

The queens QN_1 and QN_2 use the moves of rooks RK and bishops BP_1 and BP_2, respectively. It follows that

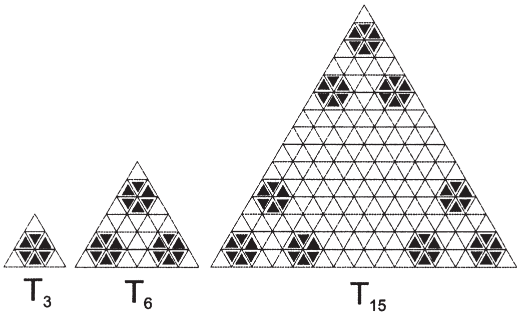

Figure 27: Gameboards T_n with $b_n = 4n - 6$ for $n = \frac{3^{k+1}+3}{2}$, $k = 0,1,2$.

$$\beta_n(QN_1) \le \beta_n(QN_2) \le \beta_n(RK) = \left\lfloor \frac{2n+1}{3} \right\rfloor .$$

For $n \le 36$ we know that

$$\beta_n(QN_1) = \begin{cases} \left\lfloor \frac{2n+1}{3} \right\rfloor - 1 & \text{if } n = 3,4,6,7,13,16,19, \\ \left\lfloor \frac{2n+1}{3} \right\rfloor & \text{otherwise.} \end{cases}$$

In general, we so far did not discover a regularity for maximum sets of independent queens.

References

[1] Bode, J.-P. & H. Harborth (2000): Independent chess pieces on Euclidean boards. J. Combin. Math. Combin. Comput. **33**: 209–223.

[2] Conway, J.H. & J.C. Lagarias (1990): Tiling with polyominoes and combinatorial group theory. J. Combin. Theory, Series A **53**: 183–208.

[3] Harborth, H. & M. Harborth (1999): Bishop and rook independence on triangle boards. Congr. Numer. **138** : 199–210.

[4] Harborth, H. & P. Stark (1997): Independent knights on triangular honeycombs. Congr. Numer. **126**: 157–161.

[5] Harborth, H. & P. Stark (1998): Kings on triangular honeycombs. Geombinatorics **7**: 117–122.